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On the hyperinvariant subspace problem III

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This paper is dedicated to the memory of our dear friends Constantin Apostol and Domingo Herrero, without whose wonderful mathematics it would not exist.

Abstract

In two recent papers (Foias and Pearcy, J. Funct. Anal., in press, Hamid et al., Indiana Univ. Math. J., to appear), the authors reduced the hyperinvariant subspace problem for operators on Hilbert space to the question whether every C_{00}-(BCP)-operator that is quasidiagonal and has spectrum the unit disc has a nontrivial hyperinvariant subspace (n.h.s.). In this note, we continue this study by showing, with the help of a new equivalence relation, that every operator whose spectrum is uncountable, as well as every nonalgebraic operator with finite spectrum, has a hyperlattice (i.e., lattice of hyperinvariant subspaces) that is isomorphic to the hyperlattice of a C_{00}, quasidiagonal, (BCP)-operator whose spectrum is the closed unit disc.

Keywords: Hyperinvariant subspace; Quasisimilar; Hyperlattice

1. Preliminaries

In this note $\mathcal{H}$ will always be a fixed separable, infinite dimensional, complex, Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. If $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$ we denote by $\mathcal{S}^-$ the norm-closure of $\mathcal{S}$. For T in $\mathcal{L}(\mathcal{H})$ we write, as usual, $\sigma(T)$, $\sigma_e(T)$, and $\sigma_{le}(T)$, for the spectrum, essential (Calkin) spectrum, and left essential spectrum

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of T, respectively. The set of all scalar multiples of $1_{\mathcal{H}}$ will be written as $\mathbb{C}1_{\mathcal{H}}$, and the closed ideal of all compact operators in $\mathcal{L}(\mathcal{H})$ as $\mathbb{K}(\mathcal{H})$. If $\mathcal{C}$ is any subset of $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{C}'$ the *commutant* of $\mathcal{C}$, i.e., $\mathcal{C}' = \{T \in \mathcal{L}(\mathcal{H}) : ST = TS$ for every S in $\mathcal{C}\}$. Recall that a subspace (closed linear manifold) $\mathcal{M} \subset \mathcal{H}$ is said to be a *nontrivial hyperinvariant subspace* (n.h.s.) for a fixed operator T in $\mathcal{L}(\mathcal{H})$ if $(0) \neq \mathcal{M} \neq \mathcal{H}$ and $S\mathcal{M} \subset \mathcal{M}$ for each S in $\{T\}'$, and that the lattice of all hyperinvariant subspaces of T (including (0) and $\mathcal{H}$) is denoted by $\mathrm{Hlat}(T)$. This lattice will frequently be called *the hyperlattice of T*, and if $\mathcal{L}_1$ and $\mathcal{L}_2$ are any two lattices, we write $\mathcal{L}_1 \equiv \mathcal{L}_2$ to signify that these lattices are *isomorphic as lattices*. The (open) *hyperinvariant subspace problem* (for operators on Hilbert space) is the question whether every operator T in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ has a n.h.s.

Recall also that a completely nonunitary (c.n.u.) contraction T in $\mathcal{L}(\mathcal{H})$ is called a (BCP)-operator if $\mathbb{D}(:= \{\zeta \in \mathbb{C} : |\zeta| < 1\}) \cap \sigma_e(T)$ is a dominating set for the unit circle $\mathbb{T} := \partial\mathbb{D}$. Moreover, the class $C_{00}(\mathcal{H})$ consists of the set of all (c.n.u.) contractions T in $\mathcal{L}(\mathcal{H})$ such that both sequences $\{T^n\}_{n\in\mathbb{N}}$ and $\{T^{*n}\}_{n\in\mathbb{N}}$ converge to zero in the strong operator topology (SOT). The class of (BCP)-operators, introduced in [6], played an important role in the highly successful theory of dual algebras of operators, and is a subset of the larger class $\mathbb{A}_{\aleph_0}$ (see, e.g., [5] for more information about the theory of dual algebras). It is well known that operators in $\mathbb{A}_{\aleph_0}$ have several good properties. For instance, every direct sum of strict contractions can be realized, up to unitary equivalence, as a compression to some semi-invariant subspace of an arbitrary operator in $\mathbb{A}_{\aleph_0}$ [4]. Moreover, the lattice $\mathrm{Lat}(T)$ of invariant subspaces of any operator T in $\mathbb{A}_{\aleph_0}$ is known to be so large that it contains a sublattice isomorphic to the lattice of all subspaces of $\mathcal{H}$ [4, Theorem 4.8], and it also contains a countably infinite family $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of cyclic invariant subspaces with the property that $\mathcal{M}_m \cap \mathcal{M}_n = (0)$ whenever $m \neq n$ [2].

Recall also that operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are said to be *quasisimilar* (notation: $T_1 \sim T_2$) if there exist quasiaffinities X and Y in $\mathcal{L}(\mathcal{H})$ (i.e., $\ker X = \ker X^* = \ker Y = \ker Y^* = (0)$) such that $T_1X = XT_2$ and $YT_1 = T_2Y$. (Observe that $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$.)

For any ordinal number n satisfying $1 \leqslant n \leqslant \omega$, we denote by $\mathcal{H}^{(n)}$ the direct sum of n copies of $\mathcal{H}$ (i.e., for $n \in \mathbb{N}$, $\mathcal{H}^{(n)} = \oplus_{1\leqslant k\leqslant n}\mathcal{H}_k$ with $\mathcal{H}_k = \mathcal{H}$ for every k, and $\mathcal{H}^{(\omega)} = \oplus_{k\in\mathbb{N}}\mathcal{H}_k$), and by $T^{(n)}$ the direct sum (ampliation) of n copies of T acting on $\mathcal{H}^{(n)}$ in the obvious fashion. Following [7], we say that S and T are *ampliation quasisimilar* (notation: $S \overset{a}{\sim} T$) if there exist ordinal numbers $1 \leqslant n_1, n_2 \leqslant \omega$ such that $S^{(n_1)} \sim T^{(n_2)}$. (Note that we have $S \overset{a}{\sim} T$ if and only if $S^{(\omega)} \sim T^{(\omega)}$.) It was shown in [7] that if $S \overset{a}{\sim} T$, then S has n.h.s. if and only if T does.

2. Hyperquasisimilarity

In this section we will introduce an equivalence relation, less general than quasisimilarity, but more general than similarity, which nevertheless preserves hyperinvariant subspace lattices. Recall that operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are called similar (notation: $T_1 \approx T_2$) if there exists an invertible $X \in \mathcal{L}(\mathcal{H})$ such that $XT_1X^{-1} = T_2$ and that if

$T_1 \approx T_2$, then $\mathrm{Hlat}(T_1) \equiv \mathrm{Hlat}(T_2)$ and $\mathrm{Lat}(T_1)$ (that is, the lattice of all *invariant* subspaces of T_1) satisfies $\mathrm{Lat}(T_1) \equiv \mathrm{Lat}(T_2)$.

Definition 2.1. A quasiaffinity Q will be said to have the *hereditary property with respect to an operator* $T \in \mathcal{L}(\mathcal{H})$ if $Q \in \{T\}'$ and $(Q\mathcal{M})^- = \mathcal{M}$ for every $\mathcal{M} \in \mathrm{Hlat}(T)$.

The following elementary lemma will be useful later. For an operator T in $\mathcal{L}(\mathcal{H})$ we write, as usual, $W(T)$ for the numerical range of T.

Lemma 2.2. *Suppose $Q \in \mathcal{L}(\mathcal{H})$ is a quasiaffinity and $0 \notin W(Q)$. Then Q has the hereditary property with respect to every $T \in \mathcal{L}(\mathcal{H})$ such that $Q \in \{T\}'$.*

Proof. If $Q \in \{T\}'$ and $\mathcal{M} \in \mathrm{Hlat}(T)$ with $(Q\mathcal{M})^- \neq \mathcal{M}$, then there exists a unit vector x in $\mathcal{M} \ominus (Q\mathcal{M})^-$ and $\langle Qx, x\rangle = 0$. □

Corollary 2.3. *Suppose $Q \in \mathcal{L}(\mathcal{H})$is a quasiaffinity and there exists $0 \leqslant \theta < 2\pi$ such that $R = \mathrm{Re}(e^{i\theta}Q)$ is positive definite (i.e., $\langle Rx, x\rangle > 0$ for every $x \neq 0$ in $\mathcal{H}$). Then Q has the hereditary property with respect to every $T \in \mathcal{L}(\mathcal{H})$ for which $Q \in \{T\}'$.*

Proof. If $\langle Qx, x\rangle = 0$, then $\langle Rx, x\rangle = \left\langle \frac{1}{2}(e^{i\theta}Q + e^{-i\theta}Q^*)x, x\right\rangle = 0$ so $x = 0$. □

We are now ready for the definition of an equivalence relation that will be very useful in what follows and may be new.

Definition 2.4. Suppose that $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and are quasisimilar, so there exist quasiaffinities X and Y such that $T_1X = XT_2$ and $YT_1 = T_2Y$ (which implies, as noted above, that $XY \in \{T_1\}'$, $YX \in \{T_2\}'$). If there exists an implementing pair (X, Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2, then we say that T_1 is *hyperquasisimilar* to T_2 (notation: $T_1 \overset{h}{\sim} T_2$).

In Proposition 2.6 below we will show that $\overset{h}{\sim}$ is an equivalence relation, but first we establish another important property of the relation.

Theorem 2.5. *Suppose $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and $T_1 \overset{h}{\sim} T_2$, with X and Y as in Definition 2.4. Then the map J_1 given by $J_1(\mathcal{N}) := (X\mathcal{N})^-$, $\mathcal{N} \in \mathrm{Hlat}(T_2)$, is a lattice isomorphism of $\mathrm{Hlat}(T_2)$ onto $\mathrm{Hlat}(T_1)$, and the map J_2 given by $J_2(\mathcal{M}) := (X\mathcal{M})^-$, $\mathcal{M} \in \mathrm{Hlat}(T_1)$, is a lattice isomorphism of $\mathrm{Hlat}(T_1)$ onto $\mathrm{Hlat}(T_2)$. Moreover J_1 and J_2 are inverses of one another.*

Proof. We first define the two mappings $J_1 : \mathrm{Hlat}(T_2) \to \mathrm{Hlat}(T_1)$ and $J_2 : \mathrm{Hlat}(T_1) \to \mathrm{Hlat}(T_2)$ as follows, and then we show that they have the desired properties.

$$J_1(\mathcal{N}) = \bigvee \left\{A_1X\mathcal{N} : A_1 \in \{T_1\}'\right\}, \quad \mathcal{N} \in \mathrm{Hlat}(T_2),$$

$$J_2(\mathcal{M}) = \bigvee \left\{ A_2 Y \mathcal{M} : A_2 \in \{T_2\}' \right\}, \quad \mathcal{M} \in \mathrm{Hlat}(T_1). \tag{1}$$

Observe that $J_2(\mathcal{M})$ is the smallest subspace in $\mathrm{Hlat}(T_2)$ that contains the linear manifold $Y\mathcal{M}$, and, similarly, $J_1(\mathcal{N})$ is the smallest subspace in $\mathrm{Hlat}(T_1)$ that contains $X\mathcal{N}$. Note next that for all $A_1 \in \{T_1\}'$ and $A_2 \in \{T_2\}'$, we have $XA_2Y \in \{T_1\}'$ and $YA_1X \in \{T_2\}'$, so for all $\mathcal{M} \in \mathrm{Hlat}(T_1)$ and $\mathcal{N} \in \mathrm{Hlat}(T_2)$, we have $(XA_2Y)\mathcal{M} \subset \mathcal{M}$ and $(YA_1X)\mathcal{N} \subset \mathcal{N}$. It follows easily that

$$XJ_2(\mathcal{M}) \subset \mathcal{M}, YJ_1(\mathcal{N}) \subset \mathcal{N}, \mathcal{M} \in \mathrm{Hlat}(T_1), \mathcal{N} \in \mathrm{Hlat}(T_2). \tag{2}$$

Thus from (1) and (2) we get

$$XY(\mathcal{M}) \subset XJ_2(\mathcal{M}) \subset \mathcal{M}, YX(\mathcal{N}) \subset YJ_1(\mathcal{N}) \subset \mathcal{N}, \mathcal{M} \in \mathrm{Hlat}(T_1), \mathcal{N} \in \mathrm{Hlat}(T_2).$$

Since, by hypothesis, both *XY* and *YX* have the appropriate hereditary property, we obtain

$$(XJ_2(\mathcal{M}))^- = \mathcal{M}, \quad (YJ_1(\mathcal{N}))^- = \mathcal{N}, \quad \mathcal{M} \in \mathrm{Hlat}(T_1), \quad \mathcal{N} \in \mathrm{Hlat}(T_2). \tag{3}$$

Thus for all $A_1 \in \{T_1\}'$ and $A_2 \in \{T_2\}'$ we have from (3) that

$$A_1XJ_2(\mathcal{M}) \subset A_1\mathcal{M} \subset \mathcal{M}, A_2YJ_1(\mathcal{N}) \subset A_2\mathcal{N} \subset \mathcal{N}, \mathcal{M} \in \mathrm{Hlat}(T_1), \mathcal{N} \in \mathrm{Hlat}(T_2),$$

which easily gives, via (1) and (3), that

$$(J_1 \circ J_2)(\mathcal{M}) = \mathcal{M}, \quad (J_2 \circ J_1)(\mathcal{N}) = \mathcal{N}, \quad \mathcal{M} \in \mathrm{Hlat}(T_1), \quad \mathcal{N} \in \mathrm{Hlat}(T_2), \tag{4}$$

so J_1 and J_2 are bijective and inverses of one another. Moreover, if $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Hlat}(T_1)$ with $\mathcal{M}_1 \subset \mathcal{M}_2$, then obviously $J_2(\mathcal{M}_1) \subset J_2(\mathcal{M}_2)$ and similarly for J_1. It follows that J_1 and J_2 are lattice isomorphisms. Now let $\mathcal{N} \in \mathrm{Hlat}(T_2)$ and set $\mathcal{M} = J_1(\mathcal{N})$. Via (4), we get $J_2(\mathcal{M}) = \mathcal{N}$, and from (3) that $(X\mathcal{N})^- = \mathcal{M}$. Thus, $(X\mathcal{N})^- \in \mathrm{Hlat}(T_1)$ for all $\mathcal{N} \in \mathrm{Hlat}(T_2)$, and similarly $(Y\mathcal{M})^- \in \mathrm{Hlat}(T_2)$ for all $\mathcal{M} \in \mathrm{Hlat}(T_1)$. Thus, taking into account (1), we get $J_1(\mathcal{N}) = (X\mathcal{N})^-$ and $J_2(\mathcal{M}) = (Y\mathcal{M})^-$ as desired. □

Proposition 2.6. *The relation* $\overset{h}{\sim}$ *on* $\mathcal{L}(\mathcal{H})$ *is an equivalence relation.*

Proof. It is obvious that $\overset{h}{\sim}$ is reflexive and symmetric. To establish the transitivity, let $T_1 \overset{h}{\sim} T_2$ with X and Y as in Definition 2.4, and suppose that $T_2 \overset{h}{\sim} T_3$ with implementing pair (Z, W) of quasiaffinities (i.e., $T_2Z = ZT_3$, $WT_2 = T_3W$, and ZW [WZ] has the hereditary property with respect to T_2 [T_3]). Clearly, the pair (XZ, WY) of quasiaffinities implements the quasisimilarity of T_1 and T_3, so it suffices to show that $(XZ)(WY)$ and $(WY)(XZ)$ have the hereditary property with respect to T_1 and T_3, respectively.

Note that $XZWY \in \{T_1\}'$, and let $\mathcal{M}$ be arbitrary in $\mathrm{Hlat}(T_1)$. Then $(XZWY)(\mathcal{M}) \subset \mathcal{M}$, and, by symmetry, it suffices to show that $\{(XZWY)(\mathcal{M})\}^- = \mathcal{M}$. But this follows immediately from the equation

$$\{(X(ZW)(Y\mathcal{M})\}^- = (X(ZW)(Y\mathcal{M})^-)^- = (X(Y\mathcal{M})^-)^- = ((XY)(\mathcal{M}))^- = \mathcal{M},$$

since $(Y\mathcal{M})^- \in \mathrm{Hlat}(T_2)$, and the proof is complete. $\square$

The following verifies the statement made at the beginning of the section.

Proposition 2.7. *The relation* $\overset{h}{\sim}$ *is strictly weaker than similarity and strictly stronger than quasisimilarity.*

Proof. It is well known (cf. [11]) that there exist quasisimilar operators with nonisomorphic hyperlattices, which, together with Theorem 2.5, proves the second statement of the proposition. Moreover, the first statement follows from Example 2.10 below. $\square$

The main utility of Theorem 2.5 and Proposition 2.6 for present purposes is the following.

Theorem 2.8. *Suppose* $\{S_n\}_{n\in\mathbb{N}}$ *and* $\{T_n\}_{n\in\mathbb{N}}$ *are bounded sequences of operators in* $\mathcal{L}(\mathcal{H})$ *with* $\widehat{S} := \oplus_{n\in\mathbb{N}} S_n$ *and* $\widehat{T} := \oplus_{n\in\mathbb{N}} T_n$. *Suppose, moreover, that* $\{X_n\}_{n\in\mathbb{N}}$ *is a sequence of invertible operators such that*

$$X_n^{-1} S_n X_n = T_n, \quad n \in \mathbb{N}.$$

Then $\widehat{S} \overset{h}{\sim} \widehat{T}$.

Proof. As is well known, $\widehat{X} := \oplus_{n\in\mathbb{N}} X_n/\|X_n\|$ and $\widehat{Y} := \oplus_{n\in\mathbb{N}} (X_n)^{-1}/\|(X_n)^{-1}\|$ belong to $\mathcal{L}(\mathcal{H}^{(\omega)})$ and satisfy $\widehat{S}\widehat{X} = \widehat{X}\widehat{T}$, $\widehat{Y}\widehat{S} = \widehat{T}\widehat{Y}$. Moreover

$$\widehat{X}\widehat{Y} = \bigoplus_{n\in\mathbb{N}} 1/(\|X_n\|\|(X_n)^{-1}\|) = \widehat{Y}\widehat{X}$$

is a positive definite operator, and the fact that $\widehat{X}\widehat{Y}$ and $\widehat{Y}\widehat{X}$ have the appropriate hereditary properties is immediate from Corollary 2.3. $\square$

A special case of this result shows that the constructions in [7,10] actually preserve the hyperlattice (We return to this topic in Sections 4 and 5.)

Corollary 2.9. *Suppose* $T \in \mathcal{L}(\mathcal{H})$ *and there exists a sequence* $\{X_n\}_{n\in\mathbb{N}}$ *of invertible operators in* $\mathcal{L}(\mathcal{H})$ *such that the sequence* $\{X_n^{-1} T X_n\}_{n\in\mathbb{N}}$ *is bounded, and set* $\widehat{T} := \oplus_{n\in\mathbb{N}} X_n^{-1} T X_n \in \mathcal{L}(\mathcal{H}^{(\omega)})$. *Then* $\mathrm{Hlat}(\widehat{T}) \equiv \mathrm{Hlat}(T)$.

Proof. It is well known that $\mathrm{Hlat}(T^{(\omega)}) = \{\mathcal{M}^{(\omega)} : \mathcal{M} \in \mathrm{Hlat}(T)\}$ and thus that $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(T^{(\omega)})$. The result is now immediate from Theorem 2.8. $\square$

An early example of two quasisimilar operators with very different spectrum is the following from [14].

Example 2.10. (Hoover). For every $n \in \mathbb{N}\backslash\{1\}$ let S_n be an operator on an n-dimensional complex Hilbert space whose matrix with respect to some orthonormal basis for the space is one Jordan block J_n satisfying $(J_n)^n = 0$ and $(J_n)^{n-1} \neq 0$, and define

$$\widehat{S} = \bigoplus_{n\in\mathbb{N}\backslash\{1\}} S_n, \quad \widehat{T} = \bigoplus_{n\in\mathbb{N}\backslash\{1\}} (1/n) S_n.$$

One knows that S_n is similar to T_n for each n, and thus $\widehat{S} \overset{h}{\sim} \widehat{T}$ by Theorem 2.8. Hence $\widehat{S}$ and $\widehat{T}$ have isomorphic hyperlattices (which may not have been known previously), but $\widehat{S} \notin \mathbb{K}$, $\widehat{T} \in \mathbb{K}$, $\sigma(\widehat{S}) = \mathbb{D}^-$ and $\sigma(\widehat{T}) = \{0\}$, so S and T are not similar (as is well known). This shows, in conjunction with Theorem 4.4 below, that a nonzero compact operator can have the same hyperlattice as a (BCP)-operator.

Remark 2.11. The authors here express their appreciation to Hari Bercovici for several useful conversations about the contents of this paper. In particular, he kindly pointed out to us that restricted to the class of C_0-operators with property (P) (cf. [3, p.182] for definitions) the equivalence relations quasisimilarity and hyperquasisimilarity are identical. The key point is that if $T \in C_0$ and has property (P), then every quasiaffinity $Z \in \{T\}'$ has the hereditary property with respect to T.

3. Hyposimilarity

The fact that Theorem 2.5 and Proposition 2.6 are valid led the authors to ask whether some equivalence relation, modeled on $\overset{h}{\sim}$ but stronger, might be defined so as to also preserve invariant subspace lattices. This led to the results of this section of the paper, in which we introduce another equivalence relation on $\mathcal{L}(\mathcal{H})$, weaker than similarity (i.e., with larger equivalence classes) but stronger than hyperquasisimilarity, that not only preserves the hyperlattices of operators but preserves the lattices of invariant subspaces as well. Recall that for an $A \in \mathcal{L}(\mathcal{H})$, $\mathrm{Alg}\,(A)$ denotes the WOT-closed subalgebra of $\mathcal{L}(\mathcal{H})$ generated by A and $1_{\mathcal{H}}$, and $\mathrm{Alg\,Lat}(A)$ denotes the algebra $\{B \in \mathcal{L}(\mathcal{H}) : B\mathcal{M} \subset \mathcal{M}$ for each $\mathcal{M} \in \mathrm{Lat}(A)\}$.

Definition 3.1. Suppose T_1, $T_2 \in \mathcal{L}(\mathcal{H})$ and there exists a pair (X, Y) of quasiaffinities in $\mathcal{L}(\mathcal{H})$ such that $T_1 X = X T_2$ and $Y T_1 = T_2 Y$. Suppose, moreover, that $XY \in \mathrm{Alg\,Lat}(T_1)$, $YX \in \mathrm{Alg\,Lat}(T_2)$, and that

$$(XY(\mathcal{M}))^- = \mathcal{M}, \quad (YX(\mathcal{N}))^- = \mathcal{N}, \quad \mathcal{M} \in \mathrm{Lat}(T_1), \quad \mathcal{N} \in \mathrm{Lat}(T_2). \tag{5}$$

Then we say that T_1 is *hyposimilar* to T_2 (notation: $T_1 \underset{h}{\approx} T_2$).

Theorem 3.2. *The relation* $\underset{h}{\approx}$ *is an equivalence relation on* $\mathcal{L}(\mathcal{H})$, *weaker than* $\approx$ *and stronger than* $\overset{h}{\sim}$, *with the property that if* $T_1 \underset{h}{\approx} T_2$, *then* $\operatorname{Lat}(T_1) \equiv \operatorname{Lat}(T_2)$ (*and* $\operatorname{Hlat}(T_1) \equiv \operatorname{Hlat}(T_2)$).

Proof. It is obvious that $\underset{h}{\approx}$ is (formally) weaker than $\approx$ and (formally) stronger than $\overset{h}{\sim}$, and it is also obvious that $\underset{h}{\approx}$ is reflexive and symmetric. The argument to prove transitivity and lattice preservation is very like that in Theorem 2.5. Thus, suppose T_1, T_2, X, and Y are as in Definition 3.1 with $T_1 \underset{h}{\approx} T_2$, and define the mappings $G_1 : \operatorname{Lat}(T_2) \to \operatorname{Lat}(T_1)$ and $G_2 : \operatorname{Lat}(T_1) \to \operatorname{Lat}(T_2)$ by

$$G_2(\mathcal{M}) = (Y\mathcal{M})^-, \quad G_1(\mathcal{N}) = (X\mathcal{N})^-, \quad \mathcal{M} \in \operatorname{Lat}(T_1), \quad \mathcal{N} \in \operatorname{Lat}(T_2)$$

(Here it is clear that $(Y\mathcal{M})^- \in \operatorname{Lat}(T_2)$ and that $(X\mathcal{N})^- \in \operatorname{Lat}(T_1)$.) From (5) we get that

$$\{(XY)(\mathcal{M})\}^- = \mathcal{M}, \quad \{(YX)(\mathcal{N})\}^- = \mathcal{N}.$$

Hence $G_1 \circ G_2 = 1_{\operatorname{Lat}(T_1)}$ and $G_2 \circ G_1 = 1_{\operatorname{Lat}(T_2)}$, which, together with the fact that if $\mathcal{M}_1 \subset \mathcal{M}_2$ in $\operatorname{Lat}(T_1)$, then $G_2(\mathcal{M}_1) \subset G_2(\mathcal{M}_2)$, shows that G_2 is an isomorphism of $\operatorname{Lat}(T_1)$ onto $\operatorname{Lat}(T_2)$ whose inverse is G_1. The transitivity of $\underset{h}{\approx}$ is now established by an argument very like the corresponding argument in Proposition 2.6, so we say no more about it. □

Remark 3.3. It turns out that the relation $\underset{h}{\approx}$ is not the first equivalence relation to appear that is more general than similarity and preserves invariant subspace lattices. After Theorem 3.2 was proved, the authors became aware of [14], in which Kapustin introduced an equivalence relation called pseudosimilarity, (formally) stronger than $\underset{h}{\approx}$, which has this property as well as that of preserving the reflexivity of an operator. Whether $\underset{h}{\approx}$ is strictly weaker than pseudosimilarity and whether $\underset{h}{\approx}$ preserves reflexivity are yet to be determined, but in this connection we do have the following.

Proposition 3.4. *Restricted to the class of* C_0*-operators, the equivalence relations of pseudosimilarity and hyposimilarity coincide* (*cf.* [3,14] *for definitions*).

Proof. Since hyposimilarity is weaker by definition, it suffices to show that if T_1, $T_2 \in C_0$ and $T_1 \underset{h}{\approx} T_2$, then T_1 is pseudosimilar to T_2. Thus, suppose X and Y are quasiaffinities such that $T_1X = XT_2$, $YT_1 = T_2Y$, $XY \in \operatorname{Alg}\operatorname{Lat}(T_1)$ and $YX \in \operatorname{Alg}\operatorname{Lat}(T_2)$. Since $XY \in \{T_1\}'$ and $\operatorname{Alg}\operatorname{Lat}(T_1) \cap \{T_1\}' = \operatorname{Alg}(T_1)$, we conclude that $XY \in \operatorname{Alg}(T_1)$. Moreover, one knows that there exist H^∞-functions u and v such that $XY = (u/v)(T_1)$, where v and $m = m_{T_1}$ are relatively prime [3, Chapter 4, Corollary 1.6], and therefore $v(T_1)$ is a quasiaffinity. Hence $X_1 := v(T_1)X$ is a quasi-affinity and $T_1X_1 = X_1T_2$. It follows easily that $X_1Y = u(T_1)$ and $YX_1 = u(T_2)$. Since these operators are quasi-affinities, it follows that u and m are relatively prime. According to the H^∞-version of Beurling's Theorem (cf. [8, Chapter V, Theorem 6.3]), any weak*-closed ideal in H^∞ is of the form φH^∞ with φ inner. Consider now the ideal in H^∞ generated by u and m and its weak*-closure $\mathcal{J}(m,u)$. The corresponding inner function φ divides both m and u and therefore φ is a complex number of modulus one; hence $\mathcal{J}(m,u)^- = H^\infty$. This clearly implies that the ideal generated by $XY = u(T_1)$ is WOT-dense in $\mathrm{Alg}(T_1)$, and similarly for $YX = u(T_2)$. $\square$

4. Reduction to (BCP)-operators

In this section, we shall prove a theorem (Theorem 4.4) which generalizes the main theorem (Theorem 1.1) of [7] in two different ways. In the first place, a larger class of operators is dealt with. Secondly, it shows that all the operators discussed have a hyperlattice isomorphic to that of one of a certain special class of (BCP)-operators.

Definition 4.1. If $T \in \mathcal{L}(\mathcal{H})$ and μ is an isolated point in $\sigma(T)$, let $\mathcal{M}_\mu$ denote the largest subspace in $\mathrm{Lat}(T)$ such that $\sigma(T|_{\mathcal{M}_\mu}) = \{\mu\}$. (Obviously, $\mathcal{M}_\mu$ is the range of the Riesz idempotent corresponding to the separated subset $\{\mu\}$ of $\sigma(T)$.) An operator T whose spectrum either is uncountable or contains an isolated point μ such that $(T-\mu I)|_{\mathcal{M}_\mu}$ is not nilpotent will be said to have property (AHV), and in the second case, μ will be said to give rise to property (AHV).

Remark 4.2. Observe that each nonalgebraic operator with finite spectrum has property (AHV). Recall that if σ is a nonempty compact set in $\mathbb{C}$, then σ can be written as a disjoint union $\sigma = \sigma_1 \dot\cup \sigma_2$, where σ_1 is the perfect set of all condensation points of σ (notation: $\sigma_1 = \sigma^\dagger$), and σ_2 is countable. Thus if $\sigma(T)$ is uncountable, then $\sigma(T)^\dagger$ is a nonempty perfect subset of $\sigma(T)$. A typical example of an operator not having property (AHV) is a diagonalizable normal operator N whose spectrum has but only one accumulation point.

In addition to the special cases of the beautiful Apostol–Herrero–Voiculescu similarity orbit theorem [1, Theorem 9.2] that were used in [7], we will also need [12, Proposition 5.13], which has at least two completely different proofs (cf. [12, Chapter 5]).

Theorem 4.3. (Herrero). *Let $T \in \mathcal{L}(\mathcal{H})$ and suppose that N is a normal operator in $\mathcal{L}(\mathcal{H})$ such that $\sigma(N) = \sigma(T)$ and such that for every isolated point μ in $\sigma(T)$,* $\dim \mathcal{M}_\mu(T) = \dim \mathcal{M}_\mu(N)$ (*where $\mathcal{M}_\mu$ is as in Definition* 4.1). *Then*

$$N \in \mathcal{S}(T)^- := \{X^{-1}TX : X \ is \ invertible\}^-.$$

The following is one of the main results of the paper.

Theorem 4.4. *Let $0 \leqslant \theta < 1$ be arbitrarily given, set $\mathbb{A}_\theta := \{\zeta \in \mathbb{C} : \theta \leqslant |\zeta| \leqslant 1\}$, and let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})$ with property* (AHV). *Moreover, let μ be* *either an isolated point of* $\sigma(T)$ *giving rise to property* (AHV) *or an element of the nonempty perfect subset* $\sigma(T)^\dagger$ *of* $\sigma(T)$. *Let* $\delta = \delta(\theta, T)$ *and* $\gamma = \gamma(\theta, T)$ *be chosen so that* $\widetilde{T} = \delta(T + \gamma 1_{\mathcal{H}})$ *satisfies* $\delta(\mu + \gamma) = (1+\theta)/2$ (*so* $(1+\theta)/2 \in \sigma(\widetilde{T})$) *and* $\sigma(\widetilde{T}) \subset D = \{\zeta \in \mathbb{C} : |\zeta - (1+\theta)/2| < (1-\theta)/4\}$. *Then* $(\widetilde{T})^{(\omega)}$ *is hyperquasisimilar to a* (BCP)-*operator* $\widehat{T}$ *in the class* C_{00} *such that* $\sigma(\widehat{T}) = \sigma_{\mathrm{le}}(\widehat{T}) = \mathbb{A}_\theta$ *and such that* $\|\widehat{T}^{-1}\| = 1/\theta$ *whenever* $\theta > 0$. *Consequently, by Corollary* 2.9, $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(\widehat{T})$.

Proof. The proof naturally splits into two cases.

Case I: $(1+\theta)/2$ *is an isolated point of* $\sigma(\widetilde{T})$ *giving rise to property* (AHV). Let $\mathcal{M} \in \mathrm{Lat}(\widetilde{T})$ be the range of the Riesz idempotent associated with $\{(1+\theta)/2\}$, so $\widetilde{T}|_{\mathcal{M}} - ((1+\theta)/2)1_{\mathcal{M}}$ is not nilpotent. It is easy to see that $\widetilde{T}$ is similar to $\widetilde{T}_1 \oplus \widetilde{T}_2$, where $\widetilde{T}_1 = \widetilde{T}|_{\mathcal{M}}$ and $\sigma(\widetilde{T}_2) = \sigma(\widetilde{T}) \backslash \{(1+\theta)/2\}$. By a harmless change of notation (in view of Proposition 2.7), we write $\widetilde{T} = \widetilde{T}_1 \oplus \widetilde{T}_2$, and thus $(\widetilde{T})^{(\omega)} = (\widetilde{T}_1)^{(\omega)} \oplus (\widetilde{T}_2)^{(\omega)}$. Observe next that $(\widetilde{T}_1)^{(\omega)}$ satisfies hypotheses (a)–(d) and (f) of [7, Theorem 1.1] and that hypothesis (e) was never used in that proof. Thus the proof of that theorem goes through unchanged and shows that there is a C_{00}, (BCP)-operator $\widehat{T}_1$ with $\sigma(\widehat{T}_1) = \sigma_{\mathrm{le}}(\widehat{T}_1) = \mathbb{A}_\theta$ and $\|(\widehat{T}_1)^{-1}\| = 1/\theta$ when $\theta > 0$, such that $(\widetilde{T}_1)^{(\omega)} \overset{h}{\sim} \widehat{T}_1$. Moreover, since

$$\sigma((\widetilde{T}_2)^{(\omega)}) \subset \sigma(\widetilde{T}) \subset (\mathbb{A}_\theta)^{\mathrm{o}},$$

one knows from [15, Theorem 8.13] that $(\widetilde{T}_2)^{(\omega)}$ is similar to an operator $\widehat{T}_2$ satisfying $\|\widehat{T}_2\| < 1$ and $\|(\widehat{T}_2)^{-1}\| < 1/\theta$. Thus $(\widetilde{T})^{(\omega)} = (\widetilde{T}_1)^{(\omega)} \oplus (\widetilde{T}_2)^{(\omega)} \overset{h}{\sim} \widehat{T}_1 \oplus \widehat{T}_2 := \widehat{T}$, and it is easy to check that $\widehat{T}$ is a (BCP)-operator with all of the desired properties, so the proof is complete in this case.

Case II: $(1+\theta)/2$ *belongs to the nonempty subset* $\sigma(\widetilde{T})^\dagger$ *of* $\sigma(\widetilde{T})$. In this case write $\sigma(\widetilde{T})$ as the disjoint union $\sigma(\widetilde{T}) = \sigma(\widetilde{T})^\dagger \,\dot{\cup}\, \sigma_2$, where σ_2 is countable, and enumerate σ_2 as $\{\lambda_i\}_{i \in J}$ where J is some initial segment of $\mathbb{N}$. We now construct two normal operators M_1 and M_2 as follows. Let v be a finite Borel measure on $\sigma(\widetilde{T})^\dagger$ such that the measure of every relatively open Borel subset of $\sigma(\widetilde{T})^\dagger$ is positive, and define M_1 to be multiplication by the position function on $L^2(\sigma(\widetilde{T})^\dagger, v)$. Then, of course, since $\sigma(\widetilde{T})^\dagger$ is perfect, $\sigma(M_1) = \sigma_{\mathrm{lre}}(M_1) = \sigma(\widetilde{T})^\dagger$. Moreover, let M_2 be a diagonalizable normal operator whose eigenvalues are exactly the numbers $\{\lambda_i\}_{i \in J}$, with multiplicity 1 if λ_i is not an isolated point of σ_2 and with multiplicity $\dim \mathcal{M}_{\lambda_i}(\widetilde{T})$ if λ_i is isolated. It is clear that $\sigma(M_1 \oplus M_2) = \sigma(\widetilde{T})$ and that if μ is an isolated point of $\sigma(\widetilde{T})$, then (by construction) $\dim \mathcal{M}_\mu(\widetilde{T}) = \dim \mathcal{M}_\mu(M_1 \oplus M_2)$. Thus, Theorem 4.3 gives that $M_1 \oplus M_2 \in \mathcal{S}(\widetilde{T})^-$ and hence $M_1^{(\omega)} \oplus M_2^{(\omega)} \in \mathcal{S}((\widetilde{T})^{(\omega)})^-$. Since

$$(1+\theta)/2 \in \sigma(M_1^{(\omega)}) \subset D \subset (\mathbb{A}_\theta)^{\mathrm{o}}$$

and $\sigma(M_1^{(\omega)})$ is a perfect set, $M_1^{(\omega)}$ satisfies the hypotheses of [7, Theorems 3.3 and 3.4], and thus every member of the sequence $\{N_n\}_{n \in \mathbb{N}}$ of normal operators appearing in the proof of [7, Theorem 1.1] belongs to $\mathcal{S}(M_1^{(\omega)})^-$. Obviously then, $N_n \oplus M_2^{(\omega)} \in \mathcal{S}(M_1^{(\omega)} \oplus M_2^{(\omega)})^-$, and, by transitivity, $N_n \oplus M_2^{(\omega)} \in \mathcal{S}((\widetilde{T})^{(\omega)})^-$ for each $n \in \mathbb{N}$. Since $\sigma(N_n \oplus M_2^{(\omega)}) = \sigma_{\mathrm{lre}}(N_n \oplus M_2^{(\omega)}) = \sigma(N_n) = \sigma_{\mathrm{lre}}(N_n)$, the proof of Theorem 1.1 of [7] can now be carried out without change (with $N_n \oplus M_2^{(\omega)}$ replacing N_n) to produce a (BCP)-operator $\widehat{T} \overset{h}{\sim} (\widetilde{T})^{(\omega)}$ with all the desired properties. $\square$

The following corollary of Theorem 4.4 is perhaps surprising.

Corollary 4.5. *Every operator T in $\mathcal{L}(\mathcal{H})$ with singleton spectrum $\{\mu\}$ such that $T - \mu 1_{\mathcal{H}}$ is not nilpotent has a hyperlattice isomorphic to the hyperlattice of a* (BCP)-*operator with the properties enunciated in Theorem* 4.4. *In particular, this is true of every nonnilpotent quasinilpotent operator in $\mathcal{L}(\mathcal{H})$.*

Corollary 4.6. *Let $0 \leqslant \theta < 1$ be arbitrarily given. Then there exists a* (BCP)-*operator $\widehat{T}$ with the properties as in Theorem* 4.4 *such that* $\mathrm{Hlat}(\widehat{T})$ *is* (*isomorphic to*) *the interval* $[0, 1]$.

Proof. Let $S(m)$ be a Jordan block, C_0-operator in $\mathcal{L}(\mathcal{H})$ whose minimal function is $m(\lambda) = \exp\left(-\int_{\mathbb{T}} ((\zeta + \lambda)/(\zeta - \lambda))\, \mathrm{d}\nu(\zeta)\right)$, where ν is the Dirac measure at the point 1. Then, as is well known, $\mathrm{Lat}(S(m)) = \mathrm{Hlat}(S(m))$ is lattice isomorphic to the interval $[0, 1]$ and $\sigma(S(m)) = \{1\}$, so Corollary 4.5 applies to give the desired $\widehat{T}$. $\square$

Proposition 4.7. *The relation $\overset{h}{\sim}$ on $\mathcal{L}(\mathcal{H})$ is strictly weaker than $\underset{h}{\approx}$.*

Proof. Let $0 < \theta < 1$, and let $S(m)$ be as in Corollary 4.6. Set $\widetilde{T} = \delta(S(m) + \gamma 1_{\mathcal{H}})$ as in Theorem 4.4, so $\sigma(\widetilde{T}) = \{(1 + \theta)/2\}$. Then, by that theorem, $T_1 := (\widetilde{T})^{(\omega)} \overset{h}{\sim} T_2$ where $T_2 \in$ (BCP). Suppose now that $T_1 \underset{h}{\approx} T_2$, and let (X, Y) be an implementing pair of quasiaffinities as in Definition 3.1. Observe next that since $\sigma(T_1) = \{(1 + \theta)/2\}$, every $\mathcal{M} \in \mathrm{Lat}(T_1)$ also belongs to $\mathrm{Lat}(T_1^{-1})$, so $T_1 \mathcal{M} = \mathcal{M}$. Now, let $\mathcal{N} \in \mathrm{Lat}(T_2)$ be such that $T_2 \mathcal{N}$ is a proper subspace of $\mathcal{N}$. (It is well known that such subspaces exist in abundance; cf. [5, Chapter V].) By Theorem 3.2, $(X\mathcal{N})^- \in \mathrm{Lat}(T_1)$, so

$$(X\mathcal{N})^- = T_1 (X\mathcal{N})^- = (T_1 X(\mathcal{N}))^- = (X T_2(\mathcal{N}))^- \tag{6}$$

and applying Y to each side of (6) gives

$$\mathcal{N} = (YX\mathcal{N})^- = (Y(X\mathcal{N})^-)^- = (Y(XT_2(\mathcal{N}))^-)^- = (YX(T_2(\mathcal{N})))^- = T_2(\mathcal{N})$$

since $T_2(\mathcal{N}) \in \mathrm{Lat}(T_2)$. But this contradicts the choice of $\mathcal{N}$. $\square$

Remark 4.8. We note here that there are some nonscalar operators T in $\mathcal{L}(\mathcal{H})$ for which the conclusion of Theorem 4.4 regarding hyperquasisimilarity is false. For example, if T is an algebraic operator and p is a polynomial such that $p(\widetilde{T}) = 0$ (with $\widetilde{T}$ as above), then $p((\widetilde{T})^{(\omega)}) = 0$. Moreover it is clear that any operator $\widehat{T}$ that is quasisimilar to $(\widetilde{T})^{(\omega)}$ also satisfies $p(\widehat{T}) = 0$, so, in particular, $(\widetilde{T})^{(\omega)}$ can only be hyperquasisimilar to algebraic operators that are annihilated by p, and thus to operators with finite spectrum. On the other hand, note that Theorem 4.4 does imply that any completely nonunitary contraction T_0 in the class C_0 such that $\sigma(T_0)$ contains a nontrivial arc on the unit circle has a hyperlattice isomorphic to the hyperlattice of a (BCP)-operator.

5. Reduction to quasidiagonal (BCP)-operators

Recall first from [9] that an operator T in $\mathcal{L}(\mathcal{H})$ is *quasidiagonal* (notation: $T \in (QD) = (QD)(\mathcal{H})$) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank projections such that $P_n \xrightarrow{SOT} 1_{\mathcal{H}}$ and $\|TP_n - P_nT\| \to 0$, and T is *block diagonal* (notation: $T \in (BD)(\mathcal{H})$) if T is (unitarily equivalent to) a countably infinite (orthogonal) direct sum of operators T_n each of which acts on a (nonzero) finite-dimensional space. If, in addition, each of the direct summands T_n satisfies $||T_n|| < 1$, then T will be called a *strictly norm decreasing* block diagonal operator. Recall also that it is known from [9] that $(QD) = (BD) + \mathbb{K}$ and that if $T \in (QD)$ and $\varepsilon > 0$ are given, then there exist $B_\varepsilon \in (BD)$ and $K_\varepsilon \in \mathbb{K}$ such that $T = B_\varepsilon + K_\varepsilon$ and $||K_\varepsilon|| < \varepsilon$. The normed ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$ will be written as $\mathcal{C}_1(\mathcal{H})$ and the corresponding trace-norm denoted by $\|\cdot\|_1$.

We can now combine the results (and proofs thereof) of [10] with those of the earlier sections of this paper to obtain some additional structure theorems for hyperlattices of operators in $\mathcal{L}(\mathcal{H})$. We first recast some results from [10] in terms of the relation $\overset{h}{\sim}$. The proofs from [10] remain unchanged, and so are omitted.

Theorem 5.1. [10, Theorem 3.1] *Suppose $T \in \text{(BCP)}(\mathcal{H})$ and B is an arbitrary strictly norm decreasing block diagonal operator. Then, for every $\varepsilon > 0$, there exist c.n.u. contractions $T_0 = T_0(\varepsilon)$ and $K_i = K_i(\varepsilon)$, $i = 1, 2$, satisfying $K_i \in \mathcal{C}_1(\mathcal{H})$ and $\|K_i\|_1 < \varepsilon$ for $i = 1, 2$, such that, if we define $\widehat{T} \in \text{(BCP)}(\mathcal{H} \bigoplus \mathcal{H})$ as the 2×2 operator matrix*

$$\widehat{T} = \begin{pmatrix} T_0 & K_1 \\ K_2 & B \end{pmatrix},$$

then

(a) $\sigma_{\mathrm{le}}(\widehat{T}) \supset \sigma_{\mathrm{le}}(T)$, $\sigma_{\mathrm{re}}(\widehat{T}) \supset \sigma_{\mathrm{re}}(T)$, *and* $\sigma(\widehat{T}) \supset \sigma(T)$,
(b) *if* $T \in C_{00}(\mathcal{H})$, *then also* $\widehat{T} \in C_{00}(\mathcal{H} \bigoplus \mathcal{H})$, *and*
(c) $T^{(\omega)} \overset{h}{\sim} \widehat{T}$, *so* $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(\widehat{T})$.

Definition 5.2. Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and set, for each $j \in \mathbb{N}$, $\mathcal{M}_j = \bigvee\{e_1, e_2, \ldots, e_j\}$. Let $\mathbb{N}$ be partitioned as $\mathbb{N} = \cup_{j\in\mathbb{N}}\mathbb{P}_j$, where for each $j \in \mathbb{N}$, $\mathbb{P}_j$ is an infinite set, and define $\mathcal{K} = \bigoplus_{m\in\mathbb{N}} \mathcal{K}_m$, where $\mathcal{K}_m = \mathcal{M}_j$ for each $m \in \mathbb{P}_j$.

Moreover, for each $j \in \mathbb{N}$, let $\mathcal{D}_j$ be a countable set of strict contractions norm-dense in the unit ball of $\mathcal{L}(\mathcal{M}_j)$, and enumerate the elements of $\mathcal{D}_j$ as $\{B_k\}_{k \in \mathbb{P}_j}$. Define now

$$B_u := \bigoplus_{k \in \mathbb{N}} B_k \in \mathcal{L}(\mathcal{K}).$$

It is clear that B_u is a C_{00}, strictly norm decreasing, block diagonal (BCP)-operator in $\mathcal{L}(\mathcal{K})$ whose point spectrum $\sigma_p(B_u)$ is dense in $\mathbb{D}$, such that $\sigma_{\text{le}}(B_u) = \mathbb{D}^-$. It is also true (cf. [10, Corollary 4.6]) that B_u is *universal* in the sense that if S is any contraction in $\mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$ is given, then there exist operators $U : \mathcal{K} \to \mathcal{H} \oplus \mathcal{H}$ and $K \in \mathbb{K}(\mathcal{K})$ with U unitary and $||K|| < \varepsilon$ such that $U(B_u + K)U^* = B_u \oplus S$.

Putting all of the above results together, we obtain this last, perhaps surprising, new theorem.

Theorem 5.3. *Let T be any operator in $\mathcal{L}(\mathcal{H})$ with property* (AHV), *let $\varepsilon > 0$, and let B_u be the block diagonal, C_{00},* (BCP)-*operator from Definition* 5.2. *Then there exists an operator $K \in \mathbb{K}$ with $||K|| < \varepsilon$ such that*

(A) $B_u + K \in C_{00} \cap (QD) \cap (\text{BCP})$,

(B) $\sigma(B_u + K) = \sigma_{\text{le}}(B_u + K) = \mathbb{D}^-$, *and*

(C) *if $\widetilde{T} = \delta(T + \gamma 1_{\mathcal{H}})$ is as in Theorem* 4.4, *then $(\widetilde{T})^{(\omega)} \overset{h}{\sim} B_u + K$, so* $\text{Hlat}(T) \equiv \text{Hlat}(B_u + K)$.

Proof. Set $\theta = 0$ and define $\widetilde{T} = \delta(T + \gamma 1_{\mathcal{H}})$ as in Theorem 4.4. An application of that theorem yields an operator $T_1 \in C_{00} \cap (\text{BCP})$ such that $\sigma(T_1) = \sigma_{\text{le}}(T_1) = \mathbb{D}^-$ and $(\widetilde{T})^{(\omega)} \overset{h}{\sim} T_1$. We now apply Theorem 5.1 with $T = T_1$ and $B = B_u$ to obtain an operator $\widehat{T_1} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ such that $\widehat{T_1} \in C_{00} \cap (\text{BCP})$, $\sigma(\widehat{T_1}) = \sigma_{\text{le}}(\widehat{T_1}) = \mathbb{D}^-$, $(T_1)^{(\omega)} \overset{h}{\sim} \widehat{T_1}$, and $\widehat{T_1}$ has the form

$$\widehat{T_1} = \begin{pmatrix} T_0 & K_1 \\ K_2 & B_u \end{pmatrix}, \tag{7}$$

where T_0 is a c.n.u. contraction, $K_1, K_2 \in \mathcal{C}_1(\mathcal{H})$, and $||K_i|| < \varepsilon/2$ for $i = 1, 2$. Since the hyperquasisimilarities occurring in the relations $(\widetilde{T})^{(\omega)} \overset{h}{\sim} T_1$ and $(T_1)^{(\omega)} \overset{h}{\sim} \widehat{T_1}$ are both of the form covered by Theorem 2.8, we conclude easily that $((\widetilde{T})^{(\omega)})^{(\omega)} \overset{h}{\sim} \widehat{T_1}$, and since $((\widetilde{T})^{(\omega)})^{(\omega)}$ is unitarily equivalent to $(\widetilde{T})^{(\omega)}$, we get that $(\widetilde{T})^{(\omega)} \overset{h}{\sim} \widehat{T_1}$. Moreover, if we write

$$J = \begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix},$$

then $||J||_1 < \varepsilon/2$ and from (7) we get that $\widehat{T_1} = (T_0 \oplus B_u) + J$. Next we apply the universality of B_u, as mentioned in Definition 5.2, to obtain the existence of a compact operator $\widetilde{K}$ with $||\widetilde{K}|| < \varepsilon/2$ and a unitary operator U such that

$$U(B_u + \widetilde{K})U^* = T_0 \oplus B_u$$

and therefore such that

$$U(B_u + \widetilde{K} + U^*JU)U^* = (T_0 \oplus B_u) + J = \widehat{T}_1.$$

Thus, after setting $K := \widetilde{K} + U^*JU$, we have that $||K|| < \varepsilon$ and $(\widetilde{T})^{(\omega)} \overset{h}{\sim} U^*\widehat{T}_1 U = B_u + K$, so the proof is complete. $\square$

Corollary 5.4. *With B_u as in Theorem* 5.3, *let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary operator with property* (AHV), *and let $\varepsilon > 0$ be arbitrary. Then there exists $K = K(T, \varepsilon) \in \mathbb{K}$ such that* $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(B_u + K)$.

Remark 5.5. The question of what structures are possible for $\mathrm{Hlat}(T)$ if $\sigma(T)$ is countably infinite but T does not have property (AHV) will be discussed in a subsequent paper. Note also that it may well be possible to replace the class $\{B_u + K : K \in \mathbb{K}\}$ appearing in Theorem 5.3 by a subclass with even better properties.

References

[1] C. Apostol, L. Fialkow, D. Herrero, D. Voiculescu, Approximation of Hilbert space operators, vol. II. Research Notes in Mathematics, vol. 102, Pitman (Advanced Publishing Program), Boston, MA, 1984.

[2] H. Bercovici, A note on disjoint invariant subspaces, Michigan Math. J. 34 (1987) 435–439.

[3] H. Bercovici, Operator theory and arithmetic in H^∞, Mathematical Surveys and Monographs, vol. 26, American Mathematical Society, Providence, RI, 1988.

[4] H. Bercovici, C. Foias, C. Pearcy, Dilation theory and systems of simultaneous equations in the predual of an operator algebra, I, Michigan Math. J. 30 (1983) 335–354.

[5] H. Bercovici, C. Foias, C. Pearcy, Dual algebras with applications to invariant subspaces and dilation theory, CBMS Regional Conference Series in Mathematics, No. 56, American Mathematical Society, Providence, RI, 1985.

[6] S. Brown, B. Chevreau, C. Pearcy, Contractions with rich spectrum have invariant subspaces, J. Operator Theory 1 (1) (1979) 123–136.

[7] C. Foias, C. Pearcy, On the hyperinvariant subspace problem, J. Funct. Anal., accepted for publication.

[8] T. Gamelin, Uniform Algebras, Prentice-Hall, Englewood Cliffs, NJ, 1969.

[9] P. Halmos, Ten problems in Hilbert space, Bull. Amer. Math. Soc. 76 (1970) 887–933.

[10] S. Hamid, C. Onica, C. Pearcy, On the hyperinvariant subspace problem II, Indiana Univ. Math. J., to appear.

[11] D. Herrero, Quasisimilarity does not preserve the hyperlattice, Proc. Amer. Math. Soc. 65 (1978) 80–84.

[12] D. Herrero, Approximation of Hilbert Space Operators, vol. 1, second ed., Pitman Research Notes in Mathematics Series, vol. 224, Longman Scientific & Technical, Harlow; copublished in the United States with Wiley, New York, 1989.

[13] T. Hoover, Quasisimilarity of operators, Illinois J. Math. 16 (1972) 678–686.

[14] V. Kapustin, Reflexivity of operators: general methods and a criterion for almost isometric contractions, St. Petersburg Math. J. 4 (2) (1993) 319–335.
[15] V. Paulsen, Completely Bounded Maps and Dilations, Pitman Research Notes in Mathematics Series, vol. 146, Longman Scientific & Technical, Harlow; Wiley, New York, 1986.